RIGHT AND WRONG DEFINITIONS OF A LIMIT.

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The following comparison of the correct definition of the *limit of a sequence* with three other definitions, which, although incorrect, are still occasionally to be met with in elementary text-books, may be of interest to teachers who wish to clarify their ideas not only as to what a limit is, but also as to what it is not.

The six illustrative examples will serve to show that the logical content of each of the four definitions is really different from that of each of the other three.

Definition 1. (Correct.) Suppose we have a sequence of values, u_1, u_2, u_3, . . . , progressing according to any given law; and also a constant quantity c. Then the constant c is called the *limit* of the sequence, *provided* whatever quantity k your opponent may select, *you can always find a stage in the sequence* such that for all values of u beyond this stage, the difference between u and c is less than k.

In other words, whenever your opponent selects a value of k at pleasure, you must be able to find a point in the sequence such that all the values of u beyond this point lie within the range $c - k$ to $c + k$. If, for any selected value of k, the corresponding point in the sequence *cannot* thus be found, then c is *not* the limit of the sequence.

The same definition can be expressed more briefly as follows: A constant c is called the limit of a variable u, if the difference between the constant and the variable eventually becomes and remains smaller than any pre-assigned quantity k.

Definition 2. (Wrong.) A constant c is called the "limit" of a variable u, if every change in the value of u brings it nearer to c.

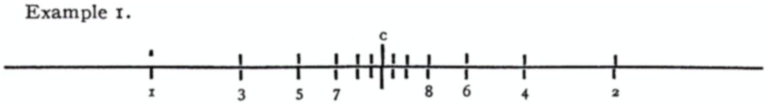
Definition 3. (Wrong.) A constant c is the "limit" of a variable u if the difference between c and u can be made less than any pre-assigned quantity, however small.

Definition 4. (Wrong.) A variable u is said to have the

"limit" c, if it continually approaches nearer and nearer to c, but never reaches c.

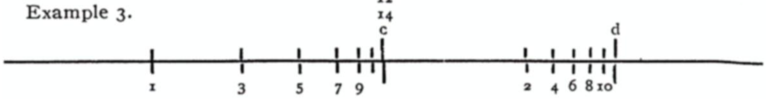
In each of the following illustrative examples, the successive values u_1, u_2 u_3, etc., are represented by points along a line, or rather by the distances to these points from a fixed origin O. The law of variation will in each case be sufficiently clear from the figure, the variable being supposed to run through the values marked 1, 2, 3, etc., in order.

Example 1.



Example 2.

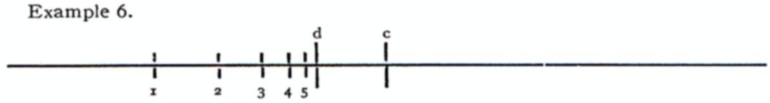
Example 3.



Example 4.

Example 5.

Example 6.



If now we inquire in each of these cases whether the constant c is the "limit" of the variable u, according to each of the four definitions, we obtain the results exhibited in the following table:

Def.	1	2	3	4	5	6
I......	Yes	Yes	No	Yes	Yes	No
II......	No	No	No	No	Yes	Yes
III......	Yes	Yes	Yes	Yes	Yes	No
IV......	Yes	No	No	No	No	Yes

It is clear from the definitions that every instance which is "yes" under definition I. will also be "yes" under definition III. With this necessary exception, the table shows that whatever two of the definitions we choose to compare, there is always an instance which is "yes" for one and "no" for the other, and also an instance which is "no" for the first and "yes" for the second. In other words, the four notions of "limit" embodied in the four definitions are absolutely distinct from one another, and should never be confused.

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